A Linear Programming Problem: Parking at the Mall

A new mall with 2 major department stores and 55 specialty shops is being built. You are a subcontractor in charge of planning and building the parking lots for the mall. The planners provide you with the following information:

- The total number of parking spaces must range from 2,000 to 2,400 spaces.
- For every employee parking space there must be at least 9 public parking spaces.
- There must be at least 20 employee parking spaces per department store and 2 employee parking spaces per specialty shop.

You anticipate that building costs will be $580 per public parking space and $600 per employee parking place.

The mall planners expect that, during an average week, revenue (average customer spending) from each public parking space will be at least $1,000 and from each employee parking space will be at least $100.

Design a proposal to present to the mall planners showing the feasible numbers of public and employee parking spaces. How many parking spaces of each type should be built to minimize the cost of building the parking lot? How many parking spaces of each type should be built to maximize weekly revenue? Use graphs, tables and technology in your proposal.
Notes

Materials:
Graphing calculator

Algebra II TEKS Focus: (2A.3) Foundations for functions. The student formulates systems of equations and inequalities from problem situations, uses a variety of methods to solve them, and analyzes the solutions in terms of the situations.

The student is expected to:

(A) analyze situations and formulate systems of equations in two or more unknowns or inequalities in two unknowns to solve problems.

(B) use algebraic methods, graphs, tables, or matrices, to solve systems of equations or inequalities.

(C) interpret and determine the reasonableness of solutions to systems of equations or inequalities for given contexts.

Scaffolding Questions:

- What are the independent variables in this situation?
- Describe the restrictions (constraints) on the independent variables.
- How will you write these restrictions algebraically?
- What do these restrictions have to do with “the feasible region?”
- How will you go about graphing these restrictions?
- What does the cost of building the parking lot depend on? What function can you write for cost?
- What does the weekly revenue (average weekly customer spending per space) depend on? What function can you write for revenue?
- What is the Corner Principle for Linear Programming?
- What representations can you use to organize your proposal and answer the questions?

Sample Solutions:

Let \( x \) = the number of public parking spaces and \( y \) = the number of employee parking spaces.

To determine the feasible number of parking spaces to build, we need to describe the constraints on \( x \) and \( y \).

Since the total number of parking spaces must be between 2,000 and 2,400,

\[
2,000 \leq x + y \leq 2,400.
\]

For every employee space there must be at least 9 public spaces, so

\[
x \geq 9y \text{ or } y \leq \frac{1}{9}x.
\]

Finally, since we need at least 20 employee spaces for each of the 2 department stores and at least 2 employee spaces
for each of the 55 specialty shops, we know that
\[ y \geq 20(2) + 2(55) \]
\[ y \geq 150. \]

The following restrictions are placed on the 2 variables:
\[ y \geq 2,000 - x \]
\[ y \leq 2,400 - x \]
\[ y \leq \frac{1}{9} x \]
\[ y \geq 150 \]

The graph of the feasible region is shown below:

(Note: A good window for a calculator graph of this feasible region is \(1,750 \leq x \leq 2,300, \ 125 \leq y \leq 275\))

The points of intersection of the boundary lines are found by solving the systems.

Point A:
\[
\begin{align*}
  x + y &= 2,000 \\
  x &= 9y
\end{align*}
\]
\[
9y + y = 2,000
\]
\[
10y = 2,000
\]
\[
y = 200
\]
\[
x = 9 \cdot 200
\]
\[
= 1,800
\]

Additional Algebra II TEKS:
(2A.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

The student is expected to:

(A) identify the mathematical domains and ranges of functions and determine reasonable domain and range values for continuous and discrete situations.

Connection to TAKS:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 3: The student will demonstrate an understanding of linear functions.

Objective 4: The student will formulate and use linear equations and inequalities.
Point B:

\[
\begin{align*}
x + y &= 2,000 \\
y &= 150
\end{align*}
\Rightarrow \begin{align*}
x + 150 &= 2,000 \\
\Rightarrow x &= 1,850 \\
y &= 150
\end{align*}
\]

Point C:

\[
\begin{align*}
x + y &= 2,400 \\
y &= 150
\end{align*}
\Rightarrow \begin{align*}
x + 150 &= 2,400 \\
\Rightarrow x &= 2,250 \\
y &= 150
\end{align*}
\]

Point D:

\[
\begin{align*}
x + y &= 2,400 \\
x &= 9y
\end{align*}
\Rightarrow \begin{align*}
9y + y &= 2,400 \\
10y &= 2,400 \\
y &= 240 \\
x &= 9 \times 240 \\
&= 2,160
\end{align*}
\]

All points in the feasible region and on the boundary of the region with integer coordinates would be feasible numbers of public and employee parking spaces.

To predict the minimum cost of building the parking lot, we need to write a cost function, \(C\). Since it costs $580 per public space and $600 per employee space, the cost function is given by

\[C = C(x, y) = 580x + 600y.\]

The Corner Principle in Linear Programming tells us that the extreme (minimum and maximum) will occur at one of the vertices of the region. The following table gives \(C\) in thousands of dollars:
To minimize the cost of building the parking lot, there should be 1,850 public spaces and 150 employee spaces. However, the difference in the cost for points A and B is only $1, so actually the selection of any point on the line segment from A to B would give a minimal cost. The cost will be about $1,163,000.

To maximize the weekly revenue, we need a weekly revenue function, $R$. Since we anticipate at least $1,000 per week per public space and $100 per week per employee space, that function is

$$R(x, y) = 1,000x + 100y$$

We apply the Corner Principle to this function, showing the weekly revenue in thousands of dollars:

<table>
<thead>
<tr>
<th>Vertex</th>
<th>$x$</th>
<th>$y$</th>
<th>$R(x,y) = 1000x + 100y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1,800</td>
<td>200</td>
<td>1,820</td>
</tr>
<tr>
<td>B</td>
<td>1,850</td>
<td>150</td>
<td>1,865</td>
</tr>
<tr>
<td>C</td>
<td>2,250</td>
<td>150</td>
<td>2,265</td>
</tr>
<tr>
<td>D</td>
<td>2,160</td>
<td>240</td>
<td>2,184</td>
</tr>
</tbody>
</table>

The weekly revenue will be maximized at $2,265,000 if the parking lot has 2,250 public spaces and 150 employee spaces.

The cost of making the parking lot may be the least at points A or B, but this is a one-time cost. However, the revenue is computed weekly. Thus, the proposal is to maximize the revenue by constructing 2,250 public spaces and 150 employee spaces.
Extension Questions:

- How can you investigate the cost of various combinations of public and employee parking spaces within the feasible region?

  You could make a table of the x- and y-coordinates of various points in the region and compute the corresponding costs.

- Would this be efficient to do?

  No. There are many points in the feasible region with integer coordinates.

- Instead of choosing points, what else might you choose?

  You could choose different values for the cost. Let $C$ be the chosen cost and investigate the equation $580x + 600y = C$.

- How would you determine $x$- and $y$-values that will satisfy the equation $580x + 600y = C$?

  We could solve for $y$ in terms of $x$ to get $y = \frac{C - 580x}{600}$. Then we could use the calculator table or graph.

- Experiment with a few different $C$ values close to the value that minimizes the cost. Graph the resulting cost equations. What do you notice about the graphs?

  The following have been graphed with the original functions:

  $$y = \frac{1,163,000 - 580x}{600} \quad y = \frac{1,192,000 - 580x}{600} \quad y = \frac{1,308,000 - 580x}{600}$$
The graphs of the cost equations are parallel lines. When the cost is the minimum, $1,163,000, the line goes through (1,850,150).

- Can you make a conjecture about the location of the vertex that minimizes cost?

  Yes. Draw a line using a fixed cost. Move it parallel to itself from right to left across the feasible region. The last vertex it passes through will minimize the cost.

- Your calculator graph looks like the line corresponding to the minimal cost is the same as the boundary line $y = 2000 - x$. Is that true?

  No. The slope of the minimal cost equation is $m = \frac{-580}{600} = -0.96$. That is so close to the slope of the boundary line, $m = -1$, it is hard to distinguish between the lines.

- What would you conjecture to be true about the location of the vertex that will maximize the weekly revenue?

  Draw a line representing a particular weekly revenue, $R$, given by $1,000x + 10y = R$. Move it parallel to itself from left to right across the feasible region. The last vertex the line passes through will maximize the revenue.